GRAPHS OF RATIONAL FUNCTIONS

A rational function is a fraction of polynomials of the type $y = \frac{g(x)}{h(x)}$ where $h(x)$ is strictly a function of x. e.g $y = \frac{5}{x+3}$, $y = \frac{x+1}{2x+6}$, $y = \frac{(x-1)(x-3)}{(x+1)(x-2)}$, $y = \frac{x^2+4x+3}{x+2}$ e.t.c

Basic Investigations

These provide information from which graphs are developed. These include;

1. Points of intersection of the curve and the axes.(x and y- intercepts).

(a) x- intercepts,

The curve cuts the x- axis when y = 0

i.e. for the curve $y = \frac{g(x)}{h(x)}$, $\frac{g(x)}{h(x)} = 0$

$\longrightarrow$ $g(x) = 0$ i.e. The numerator = 0.

Example.

Find the x –intercepts of the following curves where applicable.

(i) $y = \frac{5}{x+3}$ (ii) $y = \frac{x+1}{2x+6}$ (iii) $y = \frac{(x-1)(x-3)}{(x+1)(x-2)}$ (iv) $y = \frac{x^2+4x+3}{x+2}$ (v) $y = \frac{x^2+3}{x-1}$

Solution;

Note ;(we shall just equate the numerators to zero where applicable)

i. Since the numerator $\neq 0$ i.e equals 5 then, the curve has no x- intercepts.

ii. $x + 1 = 0$

$\longrightarrow x = -1$ Therefore the x –intercept is $(-1,0)$

iii. $(x-1)(x-3) = 0$

$\longrightarrow x = 1, x = 3$ Therefore the x –intercepts are (1,0) and (3,0)

iv. $x^2 + 4x + 3 = 0$

$(x+1)(x+3) = 0$

$\longrightarrow x = -1, x = -3$ Therefore the x –intercepts are $(-1,0)$ and $(-3,0)$

v. Since $x^2 + 3 = 0$ has **no real roots** i.e $x^2 = -3$ then the curve has no x –intercepts.

(b) y- intercepts,

The curve cuts the y- axis when x = 0

i.e. for the curve $y = \frac{g(x)}{h(x)}$,the y-intercept occurs at $y = \frac{g(0)}{h(0)}$. Implying to get the y-ordinate we just substitute x = 0 in the expression $y = \frac{g(x)}{h(x)}$.

$\longrightarrow$ the y-intercept is the point $\left(0, \frac{g(0)}{h(0)}\right)$

Example.
Find the y –intercepts of the following curves where applicable.
(i) $y = \frac{5}{x+3}$ (ii) $y = \frac{x+1}{x(2x+1)}$ (iii) $y = \frac{(x-1)(x-3)}{(x+1)(x-2)}$

Solution;
Note; (we shall just substitute x =0 in the expression given where applicable)

i. $y = \frac{5}{0+3} = \frac{5}{3}$

$\longrightarrow$ the y-intercept is the point $\left(0, \frac{5}{3}\right)$

ii. $y = \frac{0+1}{0(2(0)+1)}$. Here y is undefined since the **denominator equals zero** implying **no** y-intercept.

iii. $y = \frac{(0-1)(0-3)}{(0+1)(0-2)} = \frac{-3}{2} = -1.5$

$\longrightarrow$ the y-intercept is the point $(0, -1.5)$

2. Turning points of a curve.

These are points at which $\frac{dy}{dx} = 0$.

Examples

Find the turning points of the curves below where applicable

(i) $y = \frac{3(x-3)}{(x+1)(x-2)}$ (ii) $y = \frac{x+1}{2x+6}$

Solution

i. For $y = \frac{3(x-3)}{(x+1)(x-2)} = \frac{3x-9}{x^2-x-2}$,

For turning points, $\frac{dy}{dx} = 0$.

$\frac{dy}{dx} = \frac{(x^2-x-2).3-(3x-9).(2x-1)}{(x^2-x-2)^2}$

$\frac{3x^2-3x-6-6x^2+21x-9}{(x^2-x-2)^2} = 0$

$-3x^2 + 18x - 15 = 0$

$x^2 - 6x + 5 = 0$

$(x-1)(x-5) = 0$

$x = 1 \ , x = 5$

When $x = 1$, $y = \frac{3(1-3)}{(1+1)(1-2)} = 3$

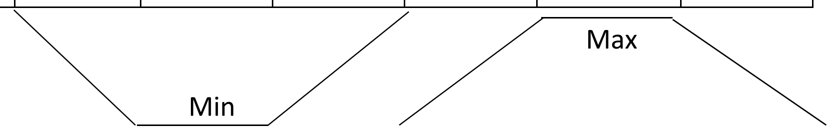
When $x = 5$, $y = \frac{3(5-3)}{(5+1)(5-2)} = \frac{1}{3}$

Therefore the turning points are (1,3) and $\left(5, \frac{1}{3}\right)$

We shall continue straight to establish the nature of the turning points above
As follows;

$$\frac{dy}{dx} = \frac{-3x^2 + 18x - 15}{(x^2 - x - 2)^2}$$

x	L	1	R	L	5	R
Sign of $\frac{dy}{dx}$	$-$	0	$+$	$+$	0	$-$



$\therefore$ The turning points are $(1,3)_{min}$ and $\left(5,\frac{1}{3}\right)_{max}$.

Note: the expression, $y = \frac{3(x-3)}{(x+1)(x-2)}$ above could have also been differentiated by introducing natural logarithms as follows.

$$ln(y) = ln\left(\frac{3(x-3)}{(x+1)(x-2)}\right)$$

$$ln(y) = ln(3x - 9) - l\,n(x + 1) - \ln(x - 2)$$

But $\boldsymbol{\frac{d}{dx}[lnf(x)] = \frac{f'(x)}{f(x)}}$**,**

So we apply that logarithmic differentiation on all the terms in our expression above as follows

$$\frac{1}{y}\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{3}{3x-9} - \frac{1}{x+1} - \frac{1}{x-2}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{3(x+1)(x-2)-(3x-9)(x-2)-(3x-9)(x+1)}{(3x-9)(x+1)(x-2)}$$

$$\frac{dy}{dx} = \frac{(-3x^2+18x-15)}{(3x-9)(x+1)(x-2)} \times \frac{(3x-9)}{(x+1)(x-2)}$$

$$\frac{dy}{dx} = \frac{-3x^2+18x-15}{[(x+1)(x-2)]^2}$$

$$\frac{dy}{dx} = \frac{-3x^2+18x-15}{(x^2-x-2)^2}$$

ii. For $y = \frac{x+1}{2x+6}$

$$\frac{dy}{dx} = \frac{(2x+6).1-(x+1).2}{(2x+6)^2} = 0$$

$$= \frac{4}{(2x+6)^2} = 0$$

$$4 = 0.(2x + 6)^2$$

Implying there is no solution. Therefore the curve has no turning points.

3. **Restricted a rea.**
   This is the region on the graph where the curve cannot exist.
   Examples
   Find the restricted region for the curve given by $y = \frac{3x-9}{x^2-x-2}$
   Solution:
   $y = \frac{3x-9}{x^2-x-2}$

   $yx^2 - x(y+3) + 9 - 2y = 0$

For non-real roots of x,

$[-(y+3)]^2 - 4y(9-2y) < 0$

$9y^2 - 30y + 9 < 0$

$3y^2 - 10y + 3 < 0$

$(3y-1)(y-3) < 0$

	$y < \frac{1}{3}$	$\frac{1}{3} < y < 3$	$y > 3$
$(3y-1)$	$-$	$+$	$+$
$(y-3)$	$-$	$-$	$+$
$(3y-1)(y-3)$	$+$	$-$	$+$

$\therefore \quad \frac{1}{3} < y < 3$

$\therefore$ The function cannot lie between $\frac{1}{3}$ and 3

4. **Asymptotes**
   An asymptote is a line to which the curve tends to approach (but actually never touches it) as one or both of the x or y values tend(s) to infinity. Asymptotes can be;
   - Vertical
   - Horizontal
   - Oblique (slanting)
   - Curved asymptote.

(a) Vertical asymptote.

A vertical asymptote is a vertical line to which the curve tends to approach (but actually never touches it) as y values tend to infinity.

To obtain vertical asymptotes, we equate the denominator of the rational function to zero and solve for x values.

Examples;

Find the vertical asymptotes for the following functions where applicable;

(i) $y = \frac{5}{x+3}$ (ii) $y = \frac{x+1}{x^2-4}$ (iii) $y = \frac{(x-1)(x-3)}{x^2-x-2}$

(iv) $y = \frac{2}{x^2-3}$ (v) $y = \frac{x^2-x-2}{x+1}$

Solution.

We shall equate the denominator to zero.

(i) $x + 3 = 0$

$x = -3$

$\therefore$ The line $x = -3$ is the vertical asymptote.

(ii) $x^2 - 4 = 0$

$x^2 = 4$

$x = \pm 2$

$\therefore$ The lines $x = -2$ and $x = 2$ are the vertical asymptotes

(iii) $x^2 - x - 2 = 0$

$(x+1)(x-2) = 0$

$x = -1, x = 2$

$\therefore$ The lines $x = -1$ and $x = 2$ are the vertical asymptotes.

(iv) $x^2 - 3 = 0$

$x^2 = 3$

$x = \pm\sqrt{3}$

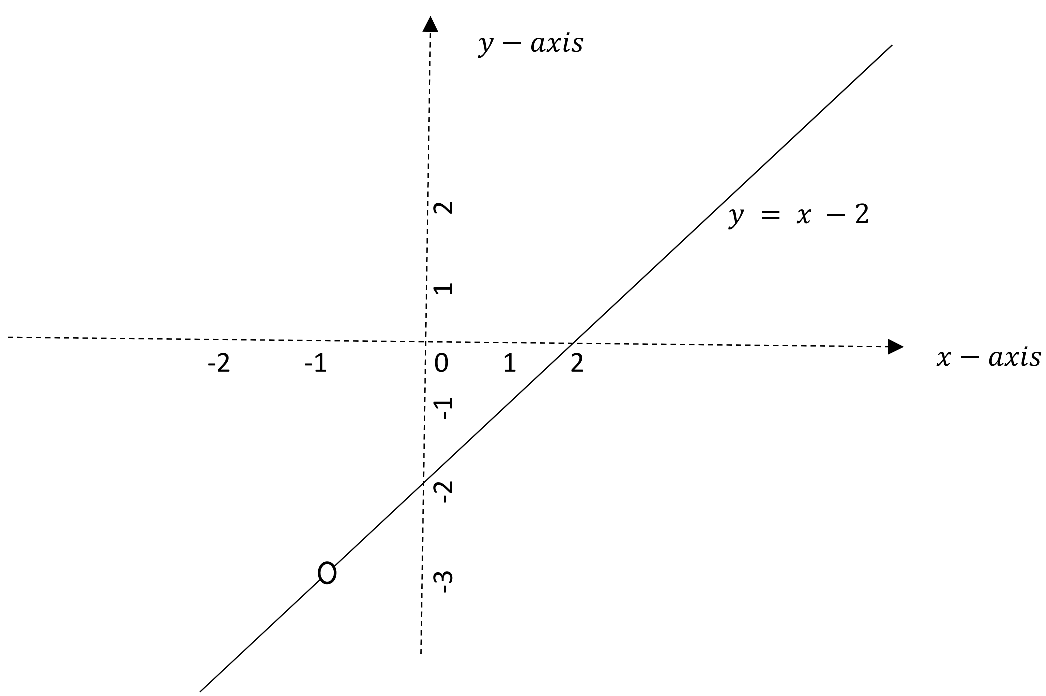
$\therefore$ The lines $x = -\sqrt{3}$ and $x = \sqrt{3}$ are the vertical asymptotes.

(v) $y = \frac{x^2-x-2}{x+1}$

$y = \frac{(x+1)(x-2)}{(x+1)}$

$y = x - 2$,which is a line. But the original function is not defined at $x = -1$ so the function is aline not defined at a point $(-1, -3)$

$\therefore$ The function has no vertical asymptote. As seen in the sketch below



(b) **Horizontal asymptote**.
A horizontal asymptote is a horizontal line to which the curve tends to approach as x values tend to $-\infty$ or $+\infty$ or both.

How to find the horizontal asymptotes for rational functions

- If the degree of the polynomial function of the numerator is less than the degree of the polynomial function of the denominator, then y = 0 (x – axis) is the horizontal asymptote of the rational function.

  Note; The degree of a polynomial is the highest power of the variable (say x or y etc.) in the individual terms of the polynomial.

  Example.
  Find the horizontal asymptote of the function $y = \frac{x+1}{(x-1)(x+2)}$
  Solution.
  We must expand first to create polynomials in the numerator and denominator to get,
  $y = \frac{x+1}{x^2+x-2}$
  The degree of the numerator is 1 and that of the denominator is 2
  Therefore the horizontal asymptote is the line y = 0 (x – axis).

  If the degree of the polynomial function of the numerator is equal to the degree of the polynomial function of the denominator, then the horizontal asymptote of the rational function is;

$$y = \frac{\text{coefficient of the term with the highest power in the numerator}}{\text{coefficient of the term with the highest power in the denominator}}.$$

  <u>**Example**</u>
  Find the horizontal asymptote of the function $y = \frac{(6x-1)(x-3)}{(3x+1)(x-2)}$
  Solution;
  The function above expands to $y = \frac{6x^2-19x+3}{3x^2-5x-2}$
  The term with the highest power in the numerator is $6x^2$ with coefficient 6 while the term with the highest power in the denominator is $3x^2$ with coefficient 3.
  The horizontal asymptote becomes $y = \frac{6}{3} = 2$
  Therefore the horizontal asymptote is the line $y = 2$.

  Note; if the degree of the numerator polynomial is greater than the degree of the denominator polynomial, then the rational function does not have a horizontal asymptote but rather has either an oblique asymptote or curved asymptote as explained below.

(c) **Oblique (slanting) asymptote**.

An oblique asymptote is a slanting line to which the curve tends to approach as x and y values tend to $-\infty$ or $+\infty$.

If the degree of the polynomial function of the numerator is greater than the degree of the polynomial function of the denominator by **1**, then the function has an oblique asymptote.

To obtain an oblique asymptote of the rational function, we first express the function in the form $y = Q(x) + \frac{R(x)}{D(x)}$ using long division.
The oblique asymptote is the line $y = Q(x)$.

Example;
Find the oblique asymptote of the curve given by $y = \frac{x^2+4x+3}{x+2}$.
Solution;
Clearly the degree of the numerator function is 2 and that of the denominator is 1 which gives a difference in degrees of 1 implying we shall have an oblique asymptote.
We shall first split the function $y = \frac{x^2+4x+3}{x-1}$ using long division;

$$
\begin{array}{r|r}
 & x+5 \\ \hline
x-1 & x^2+4x+3 \\
 & \underline{x^2-x \quad\;\;} \\
 & 5x+3 \\
 & \underline{5x-5} \\
 & 8
\end{array}
$$

This implies that $y = x + 5 + \frac{8}{x-1}$
Therefore the oblique asymptote is the line $y = x + 5$

(d) **Curved asymptote**;

A curved asymptote is a curve to which the function tends to approach as x and y values tend to $-\infty$ or $+\infty$.

If the degree of the polynomial function of the numerator is greater than the degree of the polynomial function of the denominator by **more than 1**, then the function has a curved asymptote.

To obtain a curved asymptote of the rational function, we first express the function in the form $y = Q(x) + \frac{R(x)}{D(x)}$ using long division.
The curved asymptote is the curve $y = Q(x)$.

Example;

Find the curved asymptote of the rational function $y = \frac{x^3+2}{x}$.

Solution;

Clearly the degree of the numerator function is 3 and that of the denominator is 1 which gives a difference in degrees of 2 implying we shall have a curved asymptote.

We shall first split the function $y = \frac{x^3+2}{x}$ using long division

$$\begin{array}{r|l} & \quad x^2 \\ \hline x & x^3 \quad + \; 2 \\ & \underline{x^3 \qquad\quad} \\ & \qquad\quad 2 \end{array}$$

This implies that $y = x^2 + \frac{2}{x}$

Therefore the curved asymptote is the curve $y = x^2$

(e) The behaviour(sign) of the curve $\boldsymbol{y = \frac{g(x)}{h(x)}}$ in the interval $\boldsymbol{-\infty < x < \infty}$.

To determine the sign of the function throught its domain, we construct a table.

The function, y can only change sign after the points where it cuts the x- axis and vertical asymptotes.

We shall use the points above to create regions of the curve in our table.

Examples;

Establish the nature of the function $y = \frac{(x+1)(x-6)}{(x+3)(x-2)}$. Indicating clearly its sign in the various intervals of x.

Solution;

$x -$ intercepts,

$(x+1)(x-6) = 0$

$x = -1, x = 6$

Vertical asymptotes

$(x+3)(x-2) = 0$

$x = -3, x = 2$

Region table.

	$x < -3$	$-3 < x < -1$	$-1 < x < 2$	$2 < x < 6$	$x > 6$
$(x+1)$	$-$	$-$	$+$	$+$	$+$
$(x-6)$	$-$	$-$	$-$	$-$	$+$
$(x+3)$	$-$	$+$	$+$	$+$	$+$
$(x-2)$	$-$	$-$	$-$	$+$	$+$
$y = \frac{(x+1)(x-6)}{(x+3)(x-2)}$	$+$	$-$	$+$	$-$	$+$

Note;

- In the intervals where y is +, it implies that the curve is above the x - axis
- In the intervals where y is – ,it implies that the curve is below the x – axis